Natural convection between heated vertical plates in a horizontal magnetic field

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The effect of viscous dissipation and applied magnetic field is investigated for the case of the fully developed natural convection of a fluid between two heated walls. When electrical and viscous dissipation is negligible, short-circuited Hartmann flow results. The deviation of the velocity and temperature profiles from those existing in Hartmann flow are presented for various Hartmann numbers when dissipation is not neglected. It is shown that increasing the applied magnetic field rapidly decreases the influence of both viscous and joulean dissipation on the velocity and temperature profiles.

1. Introduction

The problem of fully developed natural convection in a viscous fluid flowing between two heated vertical plates has been solved (Ostrach 1952). It is the purpose of this paper to investigate this problem when a uniform magnetic field is applied in a direction which is mutually perpendicular to the walls and the direction of flow. It is assumed that the fluid has a finite electrical conductivity and that the configuration is infinitely long in the direction of flow. The problem for two parallel plates heated to different temperatures has already been solved by Gershuni & Zhukhovitski (1958) neglecting viscous and joulean dissipation.

The fluid in the channel is subject to a buoyancy force causing it to rise and a magnetic force retarding its motion. Heat is generated within the fluid by both viscous and joulean dissipations. The motion of the fluid is determined first of all by neglecting these dissipations, and then their effect on the velocity and temperature profiles is evaluated. It is found that the magnetic field acts to decrease the dissipative effects.

2. Basic equations

The flow geometry under investigation is as shown in figure 1. A vertical channel is formed by two infinitely wide parallel plates separated by a distance $2b$. The plates are maintained at a uniform temperature T_1, which exceeds the ambient temperature T_0. A uniform magnetic field B_0 is passed across the channel normal to the plates, and a viscous conducting fluid rises in the channel driven by buoyancy forces and retarded by magnetic forces. The flow of this fluid is governed by the laws of conservation of momentum and energy. Throughout this paper the rationalized MKS system of units is used.

On the assumption that the flow is fully developed, the momentum equations in the x- and y-directions are

$$\nu\frac{\partial^2 u}{\partial y^2}-\frac{1}{\rho}\frac{\partial p}{\partial x}-g+\frac{F_x}{\rho}=0, \tag{1}$$

$$-\frac{\partial p}{\partial y}+F_y=0, \tag{2}$$

where u and p are the x-velocity and pressure of the fluid respectively, ν its kinematic viscosity, ρ its density, g the gravitational constant, and F_x and F_y components of the magnetic body force given in this geometry by

$$F_x=-jB_0, \tag{3}$$

$$F_y=jB_x. \tag{4}$$

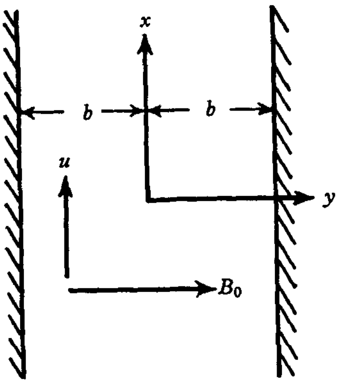

FIGURE 1. Configuration of flow.

In (3) and (4), j is the current in the z-direction given by Ohm's law

$$j=\sigma(E+uB_0), \tag{5}$$

where σ is the electrical conductivity and E the electric field. B_x is given by the Maxwell equation

$$\frac{\partial B_x}{\partial y}=-\mu j, \tag{6}$$

where μ is the permeability. If the short-circuit case is considered, E vanishes and insertion of (5) into (3) and (4) yields

$$F_x=-\sigma B_0^2 u, \tag{7}$$

$$F_y=\sigma B_0 B_x u. \tag{8}$$

In fully developed flow the pressure distribution must be hydrostatic; hence

$$\frac{\partial p}{\partial x}=-\rho_0 g, \tag{9}$$

where ρ_0 is the fluid density at ambient temperature. Inserting (7), (8) and (9) into (1) and (2), there results

$$\nu\frac{\partial^2 u}{\partial y^2}+\beta g(T-T_0)-\frac{\sigma B_0^2 u}{\rho} = 0, \tag{10}$$

$$\frac{\partial p}{\partial y} = \sigma B_0 B_x u, \tag{11}$$

where β is the expansivity of the fluid defined by

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p. \tag{12}$$

Equation (10), valid for moderate differences between T_0 and T_1, is the desired momentum relationship. Equation (11) gives the pressure gradient across the channel, which will not be considered further in this paper.

The energy equation for this flow is

$$k\frac{\partial^2 T}{\partial y^2}+\frac{j^2}{\sigma}+\rho\nu\left(\frac{\partial u}{\partial y}\right)^2 = 0, \tag{13}$$

where k is the thermal conductivity of the fluid. The second and third terms on the left represent joulean and viscous dissipations respectively. Equation (13) is the desired energy relationship.

By use of the following substitutions

$$\left.\begin{aligned} U &= \nu u[g\beta b^2(T_1-T_0)]^{-1}, \\ Y &= y/b, \\ \theta &= (T-T_0)/(T_1-T_0), \end{aligned}\right\} \tag{14}$$

equations (10) and (13) can be placed in dimensionless form, yielding

$$U''+\theta-M^2U = 0, \tag{15}$$

$$\theta''+M^2NU^2+N(U')^2 = 0, \tag{16}$$

where M is the Hartmann number, measuring the magnetic force, given by

$$M = B_0 b[\sigma/(\rho\nu)]^{\frac{1}{2}}, \tag{17}$$

and N is a dimensionless number measuring the buoyancy force given by

$$N = \rho b^4 g^2\beta^2(T_1-T_0)/(k\nu). \tag{18}$$

The primes in (15) and (16) denote differentiation with respect to Y.

Equations (15) and (16) contain two unknowns, U and θ, and must be solved simultaneously to yield the desired velocity and temperature profiles. The boundary conditions on these equations are as follows:

$$\left.\begin{aligned} U(1) &= 0, \quad \theta(1) = 1; \\ U'(0) &= 0, \quad \theta'(0) = 0. \end{aligned}\right\} \tag{19}$$

3. Solution

Due to their non-linearity, equations (15) and (16) are difficult to solve. However, the solution to a useful limiting case is readily obtainable. Allowing N to vanish is equivalent to neglecting dissipative heating, and in this case the equations become linear and have the following solutions

$$U_0 = (1 - \operatorname{sech} M \cosh MY)/M^2, \tag{20}$$

$$\theta_0 = 1, \tag{21}$$

where the zero subscript indicates that N has been set equal to zero. It can be shown that although N is not zero in practical problems, it is indeed small, which suggests the use of a perturbation technique for the solution of (15) and (16).

Eliminating θ between (15) and (16), there results

$$U^{\mathrm{iv}} - M^2 U'' = N[(U')^2 + M^2 U^2]. \tag{22}$$

Now let

$$U = U_0 + \phi N, \tag{23}$$

where the second term on the right-hand side is a correction to U_0, accounting for an N small but not zero. Substituting (23) into (22), noting that U_0 is the complimentary solution to (22), and neglecting terms in N to powers greater than unity, there results the following equation for ϕ:

$$\phi^{\mathrm{iv}} - M^2 \phi'' = (U_0')^2 + M^2 U_0^2. \tag{24}$$

This equation is readily solvable, yielding

$$\begin{aligned}\phi = {} & [(M \tanh M - \tfrac{1}{3}\cosh 2M \operatorname{sech}^2 M + 3) \operatorname{sech} M \cosh MY \\ & \times \tfrac{1}{12}(\cosh 2MY \operatorname{sech}^2 M) - \tfrac{1}{2}M^2 Y^2 - MY \sinh MY \operatorname{sech} M \\ & + \tfrac{1}{2}M^2 - 3 + \tfrac{1}{4}(\cosh 2M \operatorname{sech}^2 M)]/M^6.\end{aligned} \tag{25}$$

With ϕ now determined, equation (23) can be substituted into (15) to obtain θ. Writing θ in the form

$$\theta = \theta_0 + \epsilon N, \tag{26}$$

there results

$$\begin{aligned}\epsilon = [& -(\cosh 2MY \operatorname{sech}^2 M)/(4M^2) + 2(\cosh MY \operatorname{sech} M)/M^2 \\ & - \tfrac{1}{2}Y + \tfrac{1}{2} - 2/M^2 + (\cosh 2M \operatorname{sech}^2 M)/(4M^2)]/M^2.\end{aligned} \tag{27}$$

Equations (23) and (26) together with (25) and (27) represent the perturbation solutions for the velocity and temperature profiles, valid for sufficiently small N.

In the limiting case of zero Hartmann number, equations (25) and (27) reduce to

$$\phi = \tfrac{1}{360}(14 - 15Y^2 + Y^6), \tag{28}$$

$$\epsilon = \tfrac{1}{12}(1 - Y^4). \tag{29}$$

4. Results

The velocity and temperature profiles for the case of vanishingly small N are given by (20) and (21). The effect of N on the velocity and temperature is indicated by (23) and (26), with ϕ and ϵ given by (25) and (27) respectively.

Equations (25) and (27) are plotted in figures 2 and 3 respectively for various values of the Hartmann number. In figure 4, the values of ϕ and ϵ on the centre-line are plotted against the Hartmann number. The interesting feature of these results is that both ϕ and ϵ decrease rather rapidly with increasing M, indicating that an increase in field acts to decrease the effect of dissipation heating on the

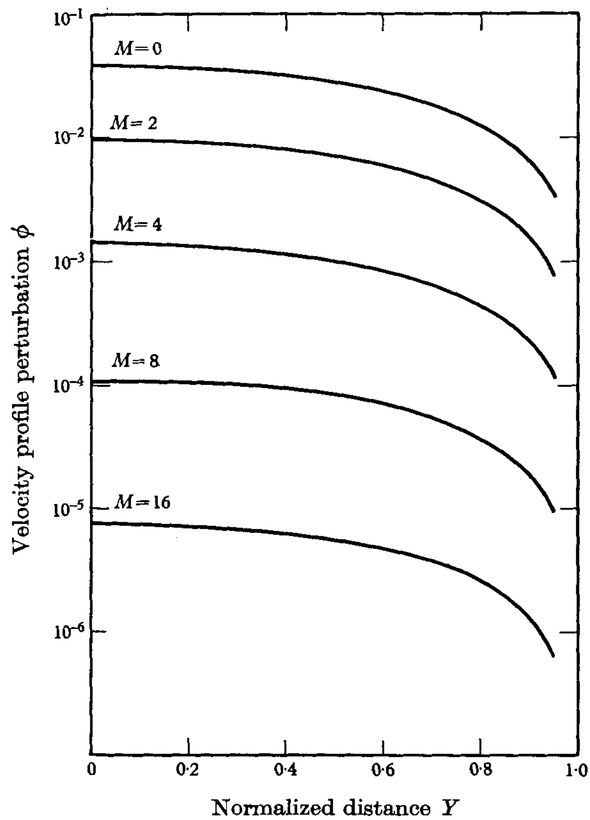

FIGURE 2. Velocity correction profiles for various Hartmann numbers.

velocity and temperature profiles. In other words, any increase in joulean dissipation accompanying an increase in field is more than overcome by the corresponding decrease in viscous dissipation. This decrease in viscous dissipation is due to the fact that the magnetic field tends to flatten the velocity profile as well as decrease the flow. From (20) it follows that the dimensionless half-channel flow rate defined by

$$Q_0 = \int_0^1 U_0 dY \tag{30}$$

for the case in which dissipations are neglected is given by

$$Q_0 = (M - \tanh M)/M^3, \tag{31}$$

which is seen to decrease with Hartmann number as expected.

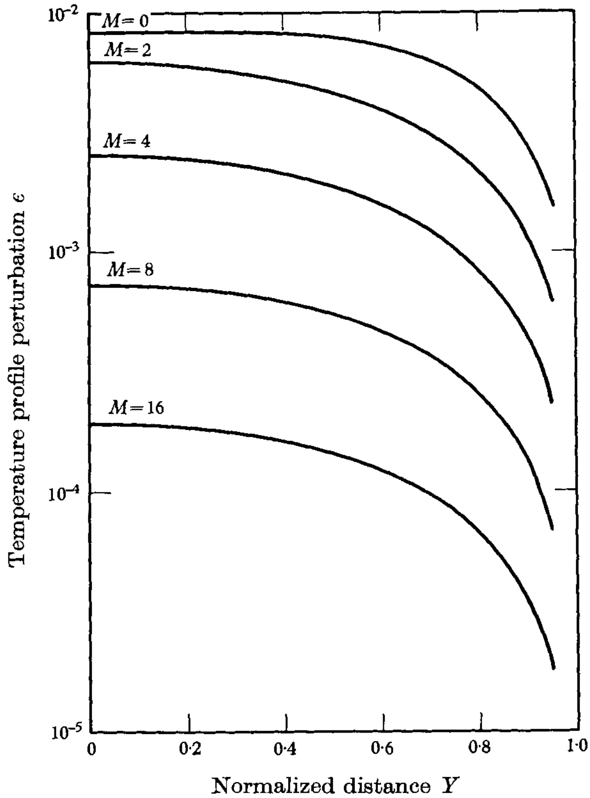

FIGURE 3. Temperature correction profiles for various Hartmann numbers.

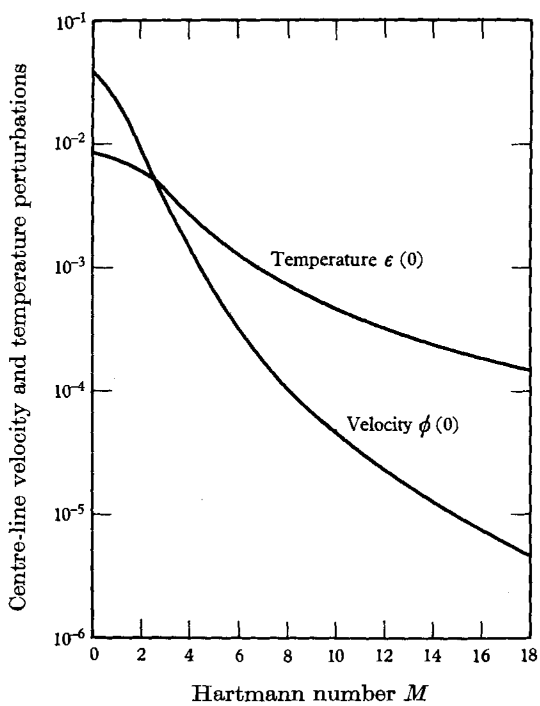

FIGURE 4. Mid-channel velocity and temperature corrections *vs* Hartmann number.

5. Numerical example

Considering the fluid to be mercury, we find that (18) yields the following expression for N as a function of the channel half-width and the temperature difference between ambient and the wall:

$$N = 4 \times 10^4 b^4 (T_1 - T_0), \tag{32}$$

where b is in metres and the temperatures in deg. C. For a channel half-width of 2 cm. and a temperature difference of 20 deg. C., N becomes 0·128. The maximum effect of dissipation is felt when the field is zero. For this case, ϕ and ϵ are given by (28) and (29). In this example the results are that the mid-channel values of U and θ are both increased about 1 % by dissipation, which in this case is all viscous. With an applied magnetic field the dissipative effects are less. For example, the 1 % increase in the mid-channel value of θ due to dissipation is reduced by a factor of 10 at a Hartmann number of 7·5, as can be seen directly from figure 4.

Since the writing of this paper, another pertinent reference has appeared in the literature (Poots 1961). In this work the open-circuited case has been treated. Our work is concerned with the short-circuited case.

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